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# ANALOGUES OF THE LAGRANGE THEOREM IN THE HYDRODYNAMICS OF WHIRLING AND STRATIFIED LIQUIDS*

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Assertions of the type of Lagrange's theorem are presented for three new classes of flows of an ideal incompressible liquid. The rest states of a liquid which is inhomogeneous with respect to its density (continuously stratified) and located in an external field of mass forces comprise the first class. Certain whirling (rotating) flows of a liquid which is homogeneous with respect to its density belong to the second and third classes. Unlike the situations which have been studied previously, flows belonging to the second and third classes are not states of relative or absolute rest and do not possess free boundaries. At the same time the formulations and proofs of the assertions are practically repeats of one another for all three cases.

The question of the existence of an analogue of Lagrange's theorem in hydrodynamics has been studied in a number of papers (/1-4/, etc.).

**1. A stratified liquid.** A closed stationary vessel is filled with an ideal incompressible liquid which is inhomogeneous with respect to its density (continuously stratified) and the vessel is located in an external field of mass forces. For simplicity, the two dimensional (planar) formulation is studied. A region $\tau$, in the plane of the Cartesian coordinates $x$ and $y$, corresponds to the interior of the vessel. A normal $\mathbf{n}$ is uniquely defined at each point of its boundary $\partial\tau$. Let $\mathbf{u}=(u, v)$ be the velocity field, and $\rho$ and $p$ be the density and pressure fields. The mass force field $\mathbf{g}(x, y)$ has a potential $U(x, y)$ such that $\mathbf{g}=-\nabla U$. The equations of motion are written in the form

$$\rho D\mathbf{u} = -\nabla p - \rho\nabla U, \quad D\rho = 0, \quad \operatorname{div}\mathbf{u} = 0 \qquad (1.1)$$
$$D \equiv \partial/\partial t + \mathbf{u}\cdot\nabla$$

The no-flow conditions are satisfied on the boundary $\partial\tau$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad (1.2)$$

The problem for (1.1), (1.2) has two first integrals

$$E = \int_\tau \rho\left(\frac{u^2+v^2}{2} + U\right)d\tau = \text{const} \qquad (1.3)$$

$$I = \int_\tau \Phi(\rho)\,d\tau = \text{const} \quad (d\tau \equiv dx\,dy) \qquad (1.4)$$

The first of these integrals has the meaning of the total energy while the second is defined in terms of an arbitrary function $\Phi(\rho)$ and is the integral expression for the invariance of the density and the volume of each fluid particle.

An approximate form of Eqs.(1.1), known as the Boussinesq approximation /5/, is also widely used. In dimensionless variables, this has the form

$$D\mathbf{u} = -\nabla p - \rho\nabla U, \quad D\rho = 0, \quad \operatorname{div}\mathbf{u} = 0 \qquad (1.5)$$

The energy integral in the case of (1.5), that is, (1.2), differs from (1.3)

$$E^* = \int_\tau \left(\frac{u^2+v^2}{2} + \rho U\right)d\tau = \text{const} \qquad (1.6)$$

The integral (1.4) remains unchanged.

The same states of hydrostatic equilibrium (rest) correspond to the cases of a stationary liquid $(\mathbf{u}=0)$ both for Eqs.(1.1) and for Eqs.(1.5)

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$$u = v = 0, \quad \rho = \rho_0(x, y), \quad p = p_0(x, y) \tag{1.7}$$

the functions $\rho_0$ and $p_0$ in which are related by the equation

$$\nabla p_0 + \rho_0 \nabla U = 0 \tag{1.8}$$

Application of the curl operator to (1.8) yields $\nabla \rho_0 \times \nabla U = 0$ which enables one to write the functional dependence $f(U, \rho_0) = 0$ or, when the conditions of the theorem on implicit functions are satisfied,

$$U = \varphi(\rho_0) \tag{1.9}$$

It is convenient to introduce the following notation. Let $\mathbf{v} = \mathbf{v}(x, y) = \nabla U / |\nabla U|$ be the field of the unit normals to the surfaces $U = \text{const}$. Then,

$$\nabla \rho = \mathbf{v}(\mathbf{v}\nabla\rho_0) \equiv \mathbf{v}\rho_{0\mathbf{v}}, \quad \mathbf{g} = g\mathbf{v}, \quad N^2 \equiv g\rho_{0\mathbf{v}} \tag{1.10}$$
$$dU/d\rho_0 = \varphi'(\rho_0) \equiv d\varphi/d\rho_0 = -(N/\rho_{0\mathbf{v}})^2$$

In the case of (1.5), the quantity $N$ has the meaning of the buoyancy frequency (the Vaisala-Brent frequency) /5/.

Now, let $u = u(x, y, t)$, $v = v(x, y, t)$, $\rho = \rho_0(x, y) + \sigma(x, y, t)$ be a certain exact non-stationary solution of problem (1.5), (1.2) which corresponds to a perturbation of the rest state (1.7).

*Assertion 1.* Let the inequality

$$0 \leqslant c^- \leqslant (N/\rho_{0\mathbf{v}})^2 \leqslant c^+ < \infty \tag{1.11}$$

with the constants $c^-$ and $c^+$ be satisfied in the whole of the region $\tau$. Then, the perturbations of $u, v$, and $\sigma$ are estimated in terms of their own initial values $u_*, v_*$ and $\sigma_*$ in the following manner:

$$\int_\tau (u^2 + v^2 + c^-\sigma^2)\, d\tau \leqslant \int_\tau (u_*^2 + v_*^2 + c^+\sigma_*^2)\, d\tau \tag{1.12}$$

*Proof.* The conservation functional

$$F(u, v, \rho) \equiv E^* + I = \int_\tau \left[\frac{u^2 + v^2}{2} + \rho U + \Phi(\rho)\right] d\tau = F(0, 0, \rho_0) + F_1 + F_2$$
$$F_1 \equiv \int_\tau \sigma\,[\varphi(\rho_0) + \Phi'(\rho_0)]\, d\tau$$
$$F_2 \equiv \int_\tau \left[\frac{u^2 + v^2}{2} + \Phi(\rho_0 + \sigma) - \Phi(\rho_0) - \Phi'(\rho_0)\,\sigma\right] d\tau$$

is made up from (1.6) and (1.4).

In $F_1$ the function $U$ is replaced in accordance with (1.9). Utilizing the arbitrary nature of $\Phi(\rho_0)$, we choose that $\Phi'(\rho_0) = -\varphi(\rho_0)$. When this is done, $F_1 \equiv 0$ and the functional $F_2$ is also found to be independent of time. Next, the inequality

$$c^- \leqslant \Phi'' \leqslant c^+ \tag{1.13}$$

which is valid in the interval of variation of $\rho_0$ in region $\tau$, follows from (1.11) and (1.10). Let the function $\Phi(\rho)$ be additionally defined for all remaining values of $\rho$ with preservation of the inequality (1.13). Then, for any number $a$,

$$\tfrac{1}{2}c^- a^2 \leqslant \Phi(\rho + a) - \Phi(\rho) - \Phi'(\rho)\, a \leqslant \tfrac{1}{2}c^+ a^2 \tag{1.14}$$

may be obtained by double integration.

The inequality (1.12) follows from the conservation of $F_2$ and (1.14).

The above proof is based on the method of a sheaf of integrals /6, 3/ in the form of /7, 8/.

The existence of an upper estimate (1.12), for an arbitrary perturbation in terms of its own initial data, means that the solution of (1.7) is stable in Lyapunov's sense /6, 3/.

Estimates of the stability can be obtained in two important cases somewhat more accurately than in the case of (1.12).

*Assertion 2.* If, in the whole of the region $\tau$,

$$0 \leqslant (N/\rho_{0\mathbf{v}})^2 = \text{const} < \infty \tag{1.15}$$

the stability of the solution of (1.7) holds for problem (1.5), (1.2) in the sense that the integral

$$\int_\tau [u^2 + v^2 + (N/\rho_{0\mathbf{v}})^2\sigma^2]\, d\tau = \text{const} \tag{1.16}$$

is independent of time.

The proof follows from the fact that, subject to condition (1.15), the quantities (1.16) and $F_2$ are identical. In the frequently encountered case of a homogeneous gravitational field $\mathbf{g} = (0, g), g = \text{const}$, conditions (1.15) are satisfied by a linear dependence $\rho_0(y)$ and a constant buoyancy frequency $N$.

*Assertion 3.* If, in the whole of the region $\tau$: $0 \leqslant N^2 < \infty$, stability occurs in the case of problem (1.5), (1.2) which has been linearized on the state (1.7) in the sense that the integral (1.16) is independent of time.

The proof is carried out by means of simple calculations.

The technique for obtaining estimates which has been presented is actually based on the existence of a variational principle. In fact, let $\delta u$, $\delta v$ and $\delta\rho$ be independent variations of the functions $u, v$ and $\rho$ which are not connected in any way whatsoever with the equations of motion. Then, the representations

$$\delta F = \int_\tau \delta\rho\,[\varphi(\rho_0) + \Phi'(\rho_0)]\,d\tau$$

$$\delta^2 F = \int_\tau [(\delta u)^2 + (\delta v)^2 + \Phi''(\rho_0)(\delta\rho)^2]\,d\tau$$

hold for the first and second variations of the functional $F = E^* + I$ at the point (1.7). The choice of $\Phi' = -\varphi$ which has already been mentioned and the inequalities (1.11) and (1.13) lead to the equality $\delta F = 0$ and the positive definiteness of $\delta^2 F$. Hence, the functional $F$ has an absolute isolated minimum in the state described by (1.7). This minimum is simultaneously a provisional minimum of the energy $E^*$ on the set of admissible functions which obey the condition $I = \text{const}$. It suffices to note that $F = E^* + \lambda I$ with a Lagrangian multiplier $\lambda = 1$. The minimum in $E^*$ can also be interpreted as a provisional minimum of potential energy in accordance with the classical formulations of the straightforward Lagrange theorem /1-4/.

All the results of Sect.1 can be transferred with only small changes to the exact problem (1.1), (1.2) concerning the motion of a stratified liquid. The two-dimensional character of the problem is also not fundamental and all the assertions are readily generalized to the three-dimensional case.

**2. Rotating flows with translational symmetry.** The motions of a liquid with a homogeneous density distribution are considered in a coordinate system which is rotating at a constant velocity $\boldsymbol{\Omega}/2$. The equations of motion are written as /3/

$$(\partial/\partial t + \mathbf{u}\cdot\nabla)\mathbf{u} + \boldsymbol{\Omega} \times \mathbf{u} = -\nabla p^*, \quad \operatorname{div}\mathbf{u} = 0 \tag{2.1}$$

($\mathbf{u}$ is the velocity vector and $p^*$ is the modified pressure which includes a "centrifugal" term).

Let $\mathbf{k}$ be a unit vector which specifies a fixed direction (in the rotating system) and makes an angle $\theta$ $(0 \leqslant \theta \leqslant \pi)$ to the vector $\boldsymbol{\Omega}$. A class of solutions of Eqs.(2.1) is studied in which $\mathbf{u}$ and $p^*$ do not change along the direcition of $\mathbf{k}$. We introduce a Cartesian coordinate system $x, y, z$ such that the $z$-axis is parallel to the vector $\mathbf{k}$, i.e. $\mathbf{k} = (0, 0, 1)$. In the case of the motions being considered, the velocity fields $\mathbf{u} = (u, v, w)$ and the pressure $p^*$ are independent of the $z$ coordinate

$$\mathbf{u} = \mathbf{u}(x, y, t), \quad p^* = p^*(x, y, t) \tag{2.2}$$

After introducing the notation

$$\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3), \quad \rho \equiv w + \Omega_1 y - \Omega_2 x \\ \mathbf{g} = (g_1, g_2, g_3) \equiv \mathbf{k} \times \boldsymbol{\Omega} = (-\Omega_2, \Omega_1, 0) \tag{2.3}$$

the system of Eqs.(2.1) for the motion (2.2) can be transformed to the form

$$Du = -p_x + \rho g_1, \quad Dv = -p_y + \rho g_2 \\ D\rho = 0, \quad u_x + v_y = 0; \quad D \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \tag{2.4}$$

Here $p \equiv p^* - \Omega_3\psi + {}^1/_2(\Omega_2 x - \Omega_1 y)^2$, and $\psi$ is the flow function for which $u = -\psi_y$, $v = \psi_x$. Everywhere, the indices on the independent variables denote partial derivatives.

If the motion (2.2) occurs in a fixed region, its boundary must have the form of a cylindrical surface with a generatrix which is parallel to the $z$-axis, i.e. it must be specified by the expression

$$f(x, y) = 0 \tag{2.5}$$

The curve (2.5) in the $xy$ plane bounds the region of flow $\tau$ which may or may not be simply connected. Its boundary $\partial\tau$, (2.5), may be closed as well as receding to infinity. The boundary conditions for no flow on (2.5) yield

$$uf_x + vf_y = 0 \tag{2.6}$$

It is a remarkable fact that Eqs.(2.4)-(2.6) are identical with the equations and the corresponding boundary conditions for the planar motions of a liquid which is inhomogeneous with respect to its density (stratified) in the Boussinesq approximation (1.5), (1.2). Hence, all the results which are valid for planar motions of a stratified liquid also simultaneously hold in the case of rotating flows with translational symmetry.

In particular, flows, which are described by the equations

$$u = v = 0, \ \rho = \rho(y_0) \tag{2.7}$$

with a coordinate $y_0$ which is read off along the direction of the vector g (2.3), will be analogues of the states of hydrostatic equilibrium (1.7). In the initial terms of (2.1), the representation (2.7) specifies the shear flow in one direction

$$u = v = 0, \quad w = w(y_0) \tag{2.8}$$

with an arbitrary function $w(y_0)$. All the results of Sect.1 can be transferred to the flows (2.7), (2.8) without any changes.

Unidirectional shear flows in the gap between two parallel rotating planes, for example, constitute a practical implementation of (2.7), (2.8). The vector $\Omega$ must be parallel to the planes while the vector g (and the $y_0$-axis) must be perpendicular to them. Assertions 1-3 of Sect.1 will specify the conditions for the stability of the flows (2.8) with respect to perturbations which are independent of the z-coordinate. The flows (2.8) will not be analogues of states of rest in the case of perturbations with other directions of invariance. In these cases the equivalence of the flows (2.8) to planar-parallel flows of a stratified liquid will hold.

**3. Flows with helical symmetry.** The motions of a liquid which is homogeneous with regard to its density in a cylindrical coordinate system $\varphi, r, z$ are described by the equations

$$\begin{gathered} Du + \frac{uv}{r} = -\frac{p_\varphi}{r}, \quad Dv - \frac{u^2}{r} = -p_r, \quad Dw = -p_z \\ v_r + \frac{v}{r} + \frac{u_\varphi}{r} + w_z = 0; \quad D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + \frac{u}{r}\frac{\partial}{\partial \varphi} + w\frac{\partial}{\partial z} \end{gathered} \tag{3.1}$$

where $u, v$, and $w$ are the $\varphi$-, $r$- and z-components of the velocity and $p$ is the pressure. In the case of motions with helical symmetry, $u, v, w$ and $p$ are functions of three independent variables $t, r$ and $\mu \equiv a\varphi - bz$.

$$p = p(t, r, \mu) \text{ and so on} \tag{3.2}$$

where $b$ is any real number and the parameter $a$ can be assumed, without loss of generality, to take just two values: 0 and 1.

When $a = 1$, all of the solutions of the form of (3.2) will be periodic with respect to $\mu$ with a period of $2\pi$ and its suffices to consider values from the interval

$$0 \leqslant \mu \leqslant 2\pi \tag{3.3}$$

When $a = 0$ (the case of rotational symmetry) the solutions may or may not be periodic. Using the notation

$$\begin{gathered} \alpha \equiv au - brw, \quad \beta \equiv bru + aw \\ R \equiv a^2 + b^2r^2, \quad g \equiv b^2r/R^2, \quad K \equiv 2ab/R^2 \end{gathered} \tag{3.4}$$

Eqs.(3.1) for the motions (3.2) may be transformed to the form

$$\begin{gathered} Dv - K\beta\alpha - (a\alpha/R)^2/r = -p_r + g\beta^2 \\ D(r\alpha/R) + K\beta rv = -p_\mu, \quad D\beta = 0 \\ v_r + \frac{v}{r} + \frac{\alpha_\mu}{r} = 0; \quad D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + \frac{\alpha}{r}\frac{\partial}{\partial \mu} \end{gathered} \tag{3.5}$$

If the motions (3.2) occur in a fixed region, its boundary must possess the required symmetry, i.e. be specified by the expression

$$f(r, \mu) = 0 \tag{3.6}$$

In the plane of the variables $r$ and $\mu$, the curve (3.6) bounds the region of flow $\tau$. The conditions of no flow for the real components of the velocity (3.1), written in terms of (3.4) and (3.6), yield

$$vf_r + (\alpha/r) f_\mu = 0 \tag{3.7}$$

Problem (3.5)-(3.7) is extremely similar to the equations and boundary conditions for planar motions of a stratified liquid witten in a polar-coordinate system. When $a = 1$, the parameter $\mu$, with the usual range of variation (3.3), plays the role of an angular variable while the parameter $\alpha$ plays the role of $\mu$, that is, of a component of the velocity. The corresponding field of the mass forces is directed along the radius. In the case of rotationally symmetric motions $(a = 0)$ the analogy passes into an equivalence. Eqs.(3.5) reduce to the form

$$Dv = -p_r + \rho g, \quad Dw = -p_z, \quad D\rho = 0 \tag{3.8}$$
$$v_r + \frac{v}{r} + w_z = 0; \quad D \equiv \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$$
$$g = r^{-3}, \quad \rho \equiv \beta^2 = (ru)^2$$

where, without any loss of generality, $b$ was chosen as being equal to -1. System (3.8) is a special case of (1.5) when the field of the mass forces is directed along a radius and there is axial symmetry.

The helical geometry of the walls described by (3.6) may be shown to be exaggerated. However, helical tubes are, in fact, presently being used in heat exchangers /9/. At the same time, a tube of circular cross-section and a pair of coaxial cylinders is an important special case of (3.6).

In the general case of problem (3.5)-(3.7), the energy integral $E$ holds in the form of a sum of a fictitious "kinetic" energy (T) and a "potential" energy ($\Pi$)

$$E = T + \Pi = \text{const} \tag{3.9}$$
$$T \equiv \frac{1}{2}\int_\tau \left(\frac{\alpha^2}{R} + v^2\right) d\tau, \quad \Pi \equiv \int_\tau \beta^2 U \, d\tau$$
$$d\tau \equiv r\,dr\,d\mu, \quad U(r) \equiv \int_0^r g(\xi)\,d\xi$$

In the initial variables of (3.1), $E$ is the kinetic energy taken in a single period. The other integral of (3.5)-(3.7) is specified by the expression

$$I = \int_\tau \Phi(\beta)\,d\tau \tag{3.10}$$

with an arbitrary function $\Phi(\beta)$. The integrals (3.9) and (3.10) are analogues of (1.6) and (1.4).

Flows described by the equations

$$v = \alpha = 0, \quad \beta = \beta_0(r) \tag{3.11}$$

where the function $\beta_0(r)$ is arbitrary, are analogues of the states of hydrostatic equilibrium for (3.5). In the initial terms of (3.1), the representation (3.11) specifies a helical flow

$$v = 0, \quad u = u_0(r), \quad w = w_0(r) \tag{3.12}$$

for which, by virtue of the fact that $\alpha = 0$, only one of the functions $u_0(r)$ or $w_0(r)$ is arbitrary while the second is determined from the relationship $au_0 = brw_0$.

Flows with circular lines of flow

$$v = w = 0, \quad u = u_0(r) \tag{3.13}$$

where the function $u_0(r)$ is arbitrarily specified, will be analogues of the states of hydrostatics in the case of the rotationally symmetric motions (3.8).

Now let $\alpha = \alpha(r, \mu, t)$, $v = v(r, \mu, t)$, $\beta = \beta(r, \mu, t)$ be a certain exact non-stationary solution of (3.5), (3.7) which is considered as a perturbation of the "rest state", (3.11), (3.12).

*Assertion 6.* Let

$$0 \leqslant c^- \leqslant g/(\beta_0^2)_r \leqslant c^+ < \infty$$

with the constants $c^-$ and $c^+$ in the whole if the region $\tau$. Then, the perturbations $\alpha, v, \sigma \equiv \beta^2 - \beta_0^2$ of the flow (3.11) are estimated in terms of its initial values $\alpha_*$, $v_*$ and $\sigma_*$ in the following manner:

$$\int_\tau \left(\frac{\alpha^2}{R} + v^2 + c^-\sigma^2\right) d\tau \leqslant \int_\tau \left(\frac{\alpha_*^2}{R} + v_*^2 + c^+\sigma_*^2\right) d\tau \tag{3.14}$$

Proof of the estimate (3.14) is based on the existence of the integrals (3.9) and (3.10). The reasoning is carried out according to a scheme which is a repeat of that used in obtaining (1.12). The analogues of assertions 2 and 3 of paragraph 1 are formulated and proved in the same way.

When the motions have rotational symmetry (i.e. in the case of problem (3.8), (3.7)), estimate (3.14) reduces to the form

$$\int (v^2 + w^2 + c^- \sigma^2)\, d\tau \leqslant \int (v_*{}^2 + w_*{}^2 + c^+ \sigma_*{}^2)\, d\tau \tag{3.15}$$

where $\sigma \equiv r^2 (u^2 - u_0{}^2)$ with $u_0(r)$ from (3.13). The constants $c^+$ and $c^-$ yield the maximum and minimum of the quantity $g/(r^2 u_0{}^2)_r$. The estimate (3.15) has a meaning if the point $r = 0$ does not belong to the region of flow (for example, in the case of flow between coaxial cylinders). Otherwise, $g/(r^2 u_0{}^2)_r \to \infty$ as $r \to 0$, a finite constant $c^+$ does not exist, and the formulation (3.15) must be changed.

The inequality (3.15) is a non-linear version of the widely known Rayleigh /10/ criterion in the linear theory of stability which guarantees the "centrifugal" stability of a flow with respect to rotationally symmetric perturbations subject to the condition that the square of the circulation $r^2 u_0{}^2$ increases as the radius $r$ increases.

*Remarks.* 1°. The estimates (1.12), (3.14), and (3.15) which have been obtained, which point to the fact that there is stability in the root mean square, may turn out to be unsatisfactory for certain purposes. In fact, if the deviations of the solutions are measured not as mean values but as the maximum values of the perturbations, it is found, as was already noted by Lyapunov /1, 2/, that the conservation of energy is insufficient to obtain assertions regarding stability. In order to obtain the corresponding estimates, it is necessary to impose additional restrictions on the solutions, and the question as to how these restrictions are to be set up remains open /1-4/.

2°. Assertions regarding the stability of helical flows are provisional in the sense that the stability of the flows (2.8), (3.12), (3.13) is only guaranteed for special classes of perturbations which possess the same symmetry as the main flows.

3°. Both examples of the reduction of the two dimensional equations of motion of a liquid which is homogeneous with regard to its density to the equations for the planar motions of a stratified liquid are based on the existence of the symmetry of helical flows and lend themselves to interpretation from the point of view of the group properties of the equations for an ideal incompressible liquid /11, 12/. The examples which have been constructed correspond to the solutions of the Euler equations which are invariant to translational and rotational transformations and combinations of them. It has been shown in /12/ that there are no other groups of invariance which do not involve a time variable in the transformation. Hence, there are, obviously, also no other cases of the reduction being considered.

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